**Elementary rules of differentiation**

Unless otherwise stated, all functions will be functions from **R** to **R**, although more generally, the formulae below make sense wherever they are [well defined](http://en.wikipedia.org/wiki/Well_defined).

**Differentiation is linear**

*Main article:* [*Linearity of differentiation*](http://en.wikipedia.org/wiki/Linearity_of_differentiation)

For any functions *f* and *g* and any real numbers *a* and *b*.

(af + bg)' = af' + bg'.\,

In other words, the derivative of the function *h*(*x*) = *a f*(*x*) + *b g*(*x*) with respect to *x* is

 h'(x) = a f'(x) + b g'(x).\, 

In [Leibniz's notation](http://en.wikipedia.org/wiki/Leibniz%27s_notation) this is written

 \frac{d(af+bg)}{dx}  = a\frac{df}{dx} +b\frac{dg}{dx}.

Special cases include:

* *The* [*constant multiple rule*](http://en.wikipedia.org/wiki/Constant_factor_rule_in_differentiation)

(af)' = a\,f' \,

* *The* [*sum rule*](http://en.wikipedia.org/wiki/Sum_rule_in_differentiation)

(f + g)' = f' + g'\,

* *The subtraction rule*

(f - g)' = f' - g'.\,

**The product or Leibniz rule**

*Main article:* [*Product rule*](http://en.wikipedia.org/wiki/Product_rule)

For any of the functions *f* and *g*,

 (fg)' = f' g + f g'.\,

In other words, the derivative of the function *h*(*x*) = *f*(*x*) \* *g*(*x*) with respect to *x* is

 h'(x) = f'(x)*g(x) + f(x)*g'(x).\, 

In Leibniz's notation this is written

\frac{d(fg)}{dx} = \frac{df}{dx} g + f \frac{dg}{dx}.

**The chain rule**

*Main article:* [*Chain rule*](http://en.wikipedia.org/wiki/Chain_rule)

This is a rule for computing the derivative of a *function of a function*, i.e., of the [composite](http://en.wikipedia.org/wiki/Function_composition)  f\circ gof two functions *f* and *g*:

(f \circ g)' = (f' \circ g)g'.\,

In other words, the derivative of the function *h*(*x*) = *f*(*g*(*x*)) with respect to *x* is

 h'(x) = f'(g(x)) g'(x).\, 

In Leibniz's notation this is written (suggestively) as:

\frac{df}{dx} = \frac{df}{dg} \frac{dg}{dx}.\,

**The polynomial or elementary power rule**

*Main article:* [*Calculus with polynomials*](http://en.wikipedia.org/wiki/Calculus_with_polynomials)

If *f*(*x*) = *xn*, for some [natural number](http://en.wikipedia.org/wiki/Natural_number) *n* (including zero) then

f'(x) = nx^{n-1}.\,

Special cases include:

* *Constant rule*: if *f* is the constant function *f*(*x*) = *c*, for any number *c*, then for all *x*

f'(x) = 0.\,

* *The derivative of a linear function is constant*: if *f*(*x*) = *ax* (or more generally, in view of the *constant rule*, if *f*(*x*)=*ax*+*b* ), then

 f'(x) = a.\,

Combining this rule with the linearity of the derivative permits the computation of the derivative of any polynomial.

**The reciprocal rule**

*Main article:* [*Reciprocal rule*](http://en.wikipedia.org/wiki/Reciprocal_rule)

For any (nonvanishing) function *f*, the derivative of the function 1/*f* (equal at *x* to 1/*f*(*x*)) is

-\frac{f'}{f^2}.\,

In other words, the derivative of *h*(*x*) = 1/*f*(*x*) is

 h'(x) = -\frac{f'(x)}{f(x)^2}.\ 

In Leibniz's notation, this is written

 \frac{d(1/f)}{dx} = -\frac{1}{f^2}\frac{df}{dx}.\,

**The inverse function rule**

*Main article:* [*inverse functions and differentiation*](http://en.wikipedia.org/wiki/Inverse_functions_and_differentiation)

This should not be confused with the reciprocal rule: the reciprocal 1/*x* of a nonzero real number *x* is its inverse with respect to multiplication, whereas the [inverse of a function](http://en.wikipedia.org/wiki/Inverse_function) is its inverse with respect to [function composition](http://en.wikipedia.org/wiki/Function_composition).

If the function *f* has an inverse *g* = *f*−1 (so that *g*(*f*(*x*)) = *x* and *f*(*g*(*y*)) = *y*) then

g' = \frac{1}{f'\circ f^{-1}}.\,

In Leibniz notation, this is written (suggestively) as

 \frac{dx}{dy} = \frac{1}{dy/dx}.

**Further rules of differentiation**

**The quotient rule**

*Main article:* [*Quotient rule*](http://en.wikipedia.org/wiki/Quotient_rule)

If *f* and *g* are functions, then:

\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}\quadwherever *g* is nonzero.

This can be derived from reciprocal rule and the product rule. Conversely (using the constant rule) the reciprocal rule is the special case *f*(*x*) = 1.

**Generalized power rule**

*Main article:* [*Power rules*](http://en.wikipedia.org/wiki/Power_rules)

The elementary power rule generalizes considerably. First, if *x* is positive, it holds when *n* is any real number. The reciprocal rule is then the special case *n* = -1 (although care must then be taken to avoid confusion with the inverse rule).

The most general power rule is the **functional power rule**: for any functions *f* and *g*,

(f^g)' = \left(e^{g\ln f}\right)' = f^g\left(f'{g \over f} + g'\ln f\right),\quad

wherever both sides are well defined.

**Logarithmic derivatives**

The [logarithmic derivative](http://en.wikipedia.org/wiki/Logarithmic_derivative) is another way of stating the rule for differentiating the logarithm of a function (using the chain rule):

 (\ln f)'= \frac{f'}{f} \quadwherever *f* is positive.

**Derivatives of exponential and logarithmic functions**

 \left(c^{ax}\right)' = {c^{ax} \ln c \cdot a } ,\qquad c > 0

note that the equation above is true for all *c*, but the derivative for c < 0 yields a complex number.

 \left(e^x\right)' = e^x

 \left( \log_c x\right)' = {1 \over x \ln c} , \qquad c > 0, c \ne 1

the equation above is also true for all *c* but yields a complex number.

 \left( \ln x\right)'  = {1 \over x} ,\qquad x \ne 0

 \left( \ln |x|\right)' = {1 \over x}

 \left( x^x \right)' = x^x(1+\ln x)

The derivative of the natural logarithm with a generalised functional argument *f*(*x*) is

 \frac{d}{dx}[\ln(f(x))] = \frac{f'(x)}{f(x)}

By applying the [change-of-base rule](http://en.wikipedia.org/wiki/Logarithmic_identities#Changing_the_base), the derivative for other bases is

\frac{d}{dx} \log_b(x) = \frac{d}{dx} \frac {\ln(x)}{\ln(b)} = \frac{1}{x \ln(b)} = \frac{\log_b(e)}{x}.

**Derivatives of trigonometric functions**

*For more details on this topic, see* [*Differentiation of trigonometric functions*](http://en.wikipedia.org/wiki/Differentiation_of_trigonometric_functions)*.*

|  |  |
| --- | --- |
| (\sin x)' = \cos x \, | (\sin^{-1} x)' = { 1 \over \sqrt{1 - x^2}} \, |
| (\cos x)' = -\sin x \, | (\cos^{-1} x)' = -{1 \over \sqrt{1 - x^2}} \, |
| (\tan x)' = \sec^2 x = { 1 \over \cos^2 x} = 1 + \tan^2 x \, | (\tan^{-1} x)' = { 1 \over 1 + x^2} \, |
| (\sec x)' = \sec x \tan x \, | (\sec^{-1} x)' = { 1 \over |x|\sqrt{x^2 - 1}} \, |
| (\csc x)' = -\csc x \cot x \, | (\csc^{-1} x)' = -{1 \over |x|\sqrt{x^2 - 1}} \, |
| (\cot x)' = -\csc^2 x = { -1 \over \sin^2 x} \, | (\cot^{-1} x)' = -{1 \over 1 + x^2} \, |

**Derivatives of hyperbolic functions**

|  |  |
| --- | --- |
| ( \sinh x )'= \cosh x = \frac{e^x +  e^{-x}}{2} | (\operatorname{arsinh}\,x)' = { 1 \over \sqrt{x^2 + 1}} |
| (\cosh x )'= \sinh x = \frac{e^x - e^{-x}}{2} | (\operatorname{arcosh}\,x)' = {\frac {1}{\sqrt{x^2-1}}} |
| (\tanh x )'= {\operatorname{sech}^2\,x} | (\operatorname{artanh}\,x)' = { 1 \over 1 - x^2} |
| (\operatorname{sech}\,x)' = - \tanh x\,\operatorname{sech}\,x | (\operatorname{arsech}\,x)' = -{1 \over x\sqrt{1 - x^2}} |
| (\operatorname{csch}\,x)' = -\,\operatorname{coth}\,x\,\operatorname{csch}\,x | (\operatorname{arcsch}\,x)' = -{1 \over |x|\sqrt{1 + x^2}} |
| (\operatorname{coth}\,x )' =   -\,\operatorname{csch}^2\,x | (\operatorname{arcoth}\,x)' = { 1 \over 1 - x^2} |

**Derivatives of special functions**

|  |  |
| --- | --- |
| [Gamma function](http://en.wikipedia.org/wiki/Gamma_function)  (\Gamma(x))' = \int_0^\infty t^{x-1} e^{-t} \ln t\,dt(\Gamma(x))' = \Gamma(x) \left(\sum_{n=1}^\infty \left(\ln\left(1 + \dfrac{1}{n}\right) - \dfrac{1}{x + n}\right) - \dfrac{1}{x}\right) = \Gamma(x) \psi(x) |  |

|  |
| --- |
| [Riemann Zeta function](http://en.wikipedia.org/wiki/Riemann_Zeta_function)  (\zeta(x))' = -\sum_{n=1}^\infty \frac{\ln n}{n^x} = -\frac{\ln 2}{2^x} - \frac{\ln 3}{3^x} - \frac{\ln 4}{4^x} - \cdots \!  (\zeta(x))' = -\sum_{p \text{ prime}} \frac{p^{-x} \ln p}{(1-p^{-x})^2}\prod_{q \text{ prime}, q \neq p} \frac{1}{1-q^{-x}} \! |

**Nth Derivatives**

The following formulae can be obtained empirically by repeated differentiation and taking notice of patterns; either by hand or computed by a [CAS (Computer Algebra System)](http://en.wikipedia.org/wiki/Computer_algebra_system).[[1]](http://en.wikipedia.org/wiki/List_of_derivatives#cite_note-0) Below *y* is the dependent variable, *x* is the independent variable, [real number](http://en.wikipedia.org/wiki/Real_number) constants are *A*, *B*, *N*, *r*, real [integers](http://en.wikipedia.org/wiki/Integers) are *n* and *j*, *F*(*x*) is a [continuously differentiable function](http://en.wikipedia.org/wiki/Continuously_differentiable_function) (the *n*th derivative exists), and *i* is the [imaginary unit](http://en.wikipedia.org/wiki/Imaginary_unit)  \sqrt{-1} \!.

|  |  |
| --- | --- |
| **Function** | ***n*th Derivative** |
| y = F(G(x)) \! | \dfrac{\mathrm{d}^n y}{\mathrm{d} x^n} = n! \displaystyle\sum_{\{k_m\}}^{} \dfrac{\mathrm{d}^r}{\mathrm{d} z^r} F(z)|_{z=G(x)} \displaystyle\prod_{m=1}^n \dfrac{1}{k_m!} \left(\dfrac{1}{m!} \dfrac{\mathrm{d}^m}{\mathrm{d} x^m} G(x) \right)^{k_m} \!  where  r = \displaystyle\sum_{m=1}^{n} k_m \!  and the set  \{k_m\} \! consists of all non-negative integer solutions of the Diophantine equation  \displaystyle\sum_{m=1}^{n} m k_m = n \!  See: Expansions for nearly Gaussian distributions by S. Blinnikov and R. Moessner [[2]](http://en.wikipedia.org/wiki/List_of_derivatives#cite_note-1) |
| y = F(x)G(x) \! | \dfrac{\mathrm{d}^n y}{\mathrm{d} x^n} = \displaystyle\sum_{k=0}^{n} \displaystyle\binom{n}{k} \dfrac{\mathrm{d}^{n-k}}{\mathrm{d} x^{n-k}} F(x) \dfrac{\mathrm{d}^k}{\mathrm{d} x^k} G(x) \! |
| y = x^N \! | \dfrac{\mathrm{d}^n y}{\mathrm{d} x^n} = \displaystyle\prod_{r=1}^n (N-r+1)x^{N-r} \! |
| y = [F(x)]^r \! | \dfrac{\mathrm{d}^n y}{\mathrm{d} x^n} = r \displaystyle\binom{n-r}{n} \displaystyle\sum_{j=0}^{n} \dfrac{(-1)^j}{r-j}{\displaystyle\binom{n}{j} [F(x)]^{r-j} \dfrac{\mathrm{d}^n}{\mathrm{d} x^n} [F(x)]^j} \! |
| y = B^{Ax} \! | \dfrac{\mathrm{d}^n y}{\mathrm{d} x^n} = A^n B^{Ax} \left ( \ln{B} \right )^n \! |
| For the case of  B = \exp(1) = e \!(*the* [exponential function](http://en.wikipedia.org/wiki/Exponential_function)),  the above reduces to:  y = e^{Ax} \! | \dfrac{\mathrm{d}^n y}{\mathrm{d} x^n} = A^n e^{Ax} \! |
| y = \ln[F(x)] \! | \dfrac{\mathrm{d}^n y}{\mathrm{d} x^n} = \delta_n \ln[F(x)] + \displaystyle\sum_{j=1}^n \dfrac{(-1)^{j-1}}{j} \binom{n}{j} \dfrac{1}{[F(x)]^j} \dfrac{\mathrm{d}^n}{\mathrm{d} x^n} [F(x)]^j \!  where  \delta_n = \begin{cases}     1 & n=0 \\     0 & n \neq 0 \\ \end{cases} \!is the [Kronecker delta](http://en.wikipedia.org/wiki/Kronecker_delta). |
| y = \sin(A x + B) \! | \dfrac{\mathrm{d}^n y}{\mathrm{d} x^n} = A^n \sin \left ( A x + B - \frac{n \pi}{2} \right ) \!  Expanding this by the [sine addition formula](http://en.wikipedia.org/wiki/Trigonometric_identity) yields a more clear form to use:  \dfrac{\mathrm{d}^n y}{\mathrm{d} x^n} = A^n \left [ \cos \left (\dfrac{n \pi}{2} \right ) \sin \left ( A x + B \right ) + \sin \left ( \dfrac{n \pi}{2} \right ) \cos \left ( A x + B \right ) \right ] \! |
| y = \cos(A x + B) \! | \dfrac{\mathrm{d}^n y}{\mathrm{d} x^n} = A^n \cos \left ( A x + B - \frac{n \pi}{2} \right ) \!  Expanding by the [cosine addition formula](http://en.wikipedia.org/wiki/Trigonometric_identity):  \dfrac{\mathrm{d}^n y}{\mathrm{d} x^n} = A^n \left [ \cos \left (\dfrac{n \pi}{2} \right ) \cos \left ( A x + B \right ) - \sin \left ( \dfrac{n \pi}{2} \right ) \sin \left ( A x + B \right ) \right ] \! |
| y = \sinh(A x + B) \! | \dfrac{\mathrm{d}^n y}{\mathrm{d} x^n} = (-i A^n) \sinh \left ( Ax + B + \dfrac{ i n \pi}{2} \right ) \! |
| y = \cosh(A x + B) \! | \dfrac{\mathrm{d}^n y}{\mathrm{d} x^n} = (\pm i A^n) \cosh \left ( Ax + B \mp \dfrac{ i n \pi}{2} \right ) \! |